Asymptotic formulas for eigenvalues and eigenfunctions of boundary value problem

O. Sh. Mukhtarov[a,*], K. Aydemir[a]

[a]*Department of Mathematics, Faculty of Arts and Science, Gaziosmanpaşa University, 60250 Tokat, Turkey*

Abstract

In this paper we are concerned with a new class of BVP' s consisting of eigendependent boundary conditions and two supplementary transmission conditions at one interior point. By modifying some techniques of classical Sturm-Liouville theory and suggesting own approaches we find asymptotic formulas for the eigenvalues and eigenfunction.

Keywords: Sturm-Liouville problems, eigenvalue, eigenfunction, asymptotics of eigenvalues and eigenfunction.

1. Introduction

Many topics in mathematical physics require investigations of eigenvalues and eigenfunctions of boundary value problems. These investigations are of utmost importance for theoretical and applied problems in mechanics, the theory of vibrations and stability, hydrodynamics, elasticity, acoustics, electrodynamics, quantum mechanics, theory of systems and their optimization, theory of random processes, and many other branches of natural science. Such problems are formulated in many different ways.

In this study we shall investigate a new class of Sturm-Liouville type problem which consist of a Sturm-Liouville equation contained

$$\Gamma(y) := -y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda) \tag{1}$$

*Corresponding Author (Tel: +90 356 252 16 16, Fax: +90 356 252 15 85)

Email addresses: `omukhtarov@yahoo.com` (O. Sh. Mukhtarov), `kadriye.aydemir@gop.edu.tr` (K. Aydemir)

to hold in finite interval $(-\pi,\pi)$ except at one inner point $0\in(-\pi,\pi)$, where discontinuity in u and u' are prescribed by transmission conditions

$$\Gamma_1(y):=a_1y'(0-,\lambda)+a_2y(0-,\lambda)+a_3y'(0+,\lambda)+a_4y(0+,\lambda)=0, \qquad (2)$$

$$\Gamma_2(y):=b_1y'(0-,\lambda)+b_2y(0-,\lambda)+b_3y'(0+,\lambda)+b_4y(0+,\lambda)=0, \qquad (3)$$

together with the boundary conditions

$$\Gamma_3(y):=\cos\alpha y(-\pi,\lambda)+\sin\alpha y'(-\pi,\lambda)=0, \qquad (4)$$

$$\Gamma_4(y):=\cos\beta y(\pi,\lambda)+\sin\beta y'(\pi,\lambda)=0. \qquad (5)$$

We describe some analytical solutions of the problem and find asymptotic formulas of eigenvalues and eigenfunctions. These boundary conditions are of great importance for theoretical and applied studies and have a definite mechanical or physical meaning (for instance, of free ends). Also the problems with transmission conditions arise in mechanics, such as thermal conduction problems for a thin laminated plate, which studied in [9]. This class of problems essentially differs from the classical case, and its investigation requires a specific approach based on the method of separation of variables. Moreover the eigenvalue parameter appear in one of the boundary conditions and two new conditions added to boundary conditions called transmission conditions.

2. The fundamental solutions and characteristic Function

Let T$=\begin{bmatrix}\beta_{10}^- & \beta_{11}^- & \beta_{10}^+ & \beta_{11}^+\\ \beta_{20}^- & \beta_{21}^- & \beta_{20}^+ & \beta_{21}^+\end{bmatrix}$. Denote the determinant of the determinant of the k-th and i-th columns of the matrix T by ρ_{kj}. Note that throughout this study we shall assume that $\rho_{12}>0$ and $\rho_{34}>0$. With a view to constructing the characteristic function we define two solution $\phi(x,\lambda)$ and $\chi(x,\lambda)$ as follows. Denote the solutions of the equation (1) satisfying the initial conditions

$$y(-\pi)=\sin\alpha,\ y'(-\pi)=-\cos\alpha \qquad (6)$$

and

$$y(\pi)=-\sin\beta,\ y'(\pi)=\cos\beta \qquad (7)$$

by $u=\phi_1(x,\lambda)$ and $u=\chi_2(x,s)$, respectively. It is known that the initial-value problems has an unique solutions $u=\phi_1(x,\lambda)$ and $u=\chi_2(x,\lambda)$, which is an entire function of $s\in\mathbb{C}$ for each fixed $x\in[-\pi,0)$ and $x\in(0,\pi]$

respectively. (see, for example, [10]). Using this solutions we can prove that the equation (1) on $[-\pi, 0)$ and $x \in (0, \pi]$ has solutions $u = \varphi^{+}(x, s)$ and $u = \psi^{-}(x, s)$, satisfying the initial conditions

$$y(0) = \frac{1}{\rho_{12}}(\rho_{23}\phi_1(0,\lambda) + \rho_{24}\frac{\partial\phi_1(0,\lambda)}{\partial x}) \tag{8}$$

$$y'(0) = \frac{-1}{\rho_{12}}(\rho_{13}\phi_1(0,\lambda) + \rho_{14}\frac{\partial\phi_1(0,\lambda)}{\partial x}). \tag{9}$$

and

$$y(0) = \frac{-1}{\rho_{34}}(\rho_{14}\chi_2(0,\lambda) + \rho_{24}\frac{\partial\chi_2(0,\lambda)}{\partial x}), \tag{10}$$

$$y'(0) = \frac{1}{\rho_{34}}(\rho_{13}\chi_2(0,\lambda) + \rho_{23}\frac{\partial\chi_2(0,\lambda)}{\partial x}). \tag{11}$$

respectively.

3. Some asymptotic approximation formulas for fundamental solutions

By applying the method of variation of parameters we can prove that the next integral and integro-differential equations are hold for $k = 0$ and $k = 1$.

$$\begin{aligned} \frac{d^k}{dx^k}\phi_1(x,\lambda) &= \sin\alpha\frac{d^k}{dx^k}\cos\left[s\left(x+\pi\right)\right] - \frac{\cos\alpha}{s}\frac{d^k}{dx^k}\sin\left[s\left(x+\pi\right)\right] \\ &+\frac{1}{s}\int\limits_{-\pi}^{x}\frac{d^k}{dx^k}\sin\left[s\left(x-z\right)\right]q(z)\phi_1(z,\lambda)dz \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{d^k}{dx^k}\chi_1(x,\lambda) &= -\frac{1}{\rho_{34}}(\rho_{14}\chi_2(0,\lambda) + \rho_{24}\frac{\partial\chi_2(0,\lambda)}{\partial x})\frac{d^k}{dx^k}\cos\left[sx\right] \\ &+\frac{1}{s\rho_{34}}(\rho_{13}\phi_2(0,\lambda) + \rho_{23}\frac{\partial\chi_2(0,\lambda)}{\partial x})\frac{d^k}{dx^k}\sin\left[sx\right] \\ &+\frac{1}{s}\int\limits_{x}^{0}\frac{d^k}{dx^k}\sin\left[s\left(x-z\right)\right]q(z)\chi_2(z,\lambda)dz \end{aligned} \tag{13}$$

for $x \in [-\pi, 0)$ and

$$\begin{aligned}\frac{d^k}{dx^k}\phi_2(x,\lambda) &= \frac{1}{\rho_{12}}(\rho_{23}\phi_1(0,\lambda)+\rho_{24}\frac{\partial\phi_1(0,\lambda)}{\partial x})\frac{d^k}{dx^k}\cos[sx] \\ &-\frac{1}{s\rho_{12}}(\rho_{13}\phi_1(0,\lambda)+\rho_{14}\frac{\partial\phi_1(0,\lambda)}{\partial x})\frac{d^k}{dx^k}\sin[sx] \\ &+\frac{1}{s}\int_0^x \frac{d^k}{dx^k}\sin[s(x-z)]\,q(z)\phi_2(z,\lambda)dz \qquad (14)\end{aligned}$$

$$\begin{aligned}\frac{d^k}{dx^k}\chi_2(x,\lambda) &= -\sin\beta\frac{d^k}{dx^k}\cos[s(\pi-x)]-\frac{\cos\beta}{s}\frac{d^k}{dx^k}\sin[s(\pi-x)] \\ &+\frac{1}{s}\int_x^\pi \frac{d^k}{dx^k}\sin[s(x-z)]\,q(z)\chi_2(z,\lambda)dz \qquad (15)\end{aligned}$$

for $x \in (0, \pi]$. Now we are ready to prove the following theorems.

Theorem 1. *Let $\lambda = s^2$, $Ims = t$. Then if $\sin\alpha \neq 0$*

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \sin\alpha\frac{d^k}{dx^k}\cos[s(x+\pi)]+O\left(|s|^{k-1}e^{|t|(x+\pi)}\right) \qquad (16)$$

$$\frac{d^k}{dx^k}\phi_{2\lambda}(x) = \frac{\rho_{24}}{\rho_{12}}\sin\alpha s\sin[s\pi]\cos[sx]+O\left(|s|^k e^{|t|(x+\pi)}\right) \qquad (17)$$

as $|\lambda| \to \infty$, while if $\sin\alpha = 0$

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = -\frac{\cos\alpha}{s}\frac{d^k}{dx^k}\sin[s(x+\pi)]+O\left(|s|^{k-2}e^{|t|(x+1)}\right) \qquad (18)$$

$$\frac{d^k}{dx^k}\phi_{2\lambda}(x) = -\frac{\rho_{24}}{\rho_{12}}\cos\alpha\cos[s\pi]\cos[sx]+O\left(|s|^{k-1}e^{|t|(x+\pi)}\right) \qquad (19)$$

as $|\lambda| \to \infty$ $(k = 0, 1)$. Each of this asymptotic equalities hold uniformly for x.

Proof.

Similarly we can easily obtain the following Theorem for $\chi_i(x,\lambda)(i = 1, 2)$.

Theorem 2. *Let* $\lambda = s^2$, $Ims = t$. *Then if* $\sin\beta \neq 0$

$$\begin{aligned} \frac{d^k}{dx^k}\chi_{2\lambda}(x) &= \sin\beta \frac{d^k}{dx^k}\cos\left[s\left(\pi - x\right)\right] + O\left(\left|s\right|^{k-1} e^{|t|(\pi - x)}\right) & (20) \\ \frac{d^k}{dx^k}\chi_{1\lambda}(x) &= -\frac{\rho_{24}}{\rho_{34}}\sin\beta s \sin\left[s\pi\right]\cos\left[sx\right] + O\left(\left|s\right|^{k} e^{|t|(\pi - x)}\right) & (21) \end{aligned}$$

as $|\lambda| \to \infty$, *while if* $\sin\beta = 0$

$$\begin{aligned} \frac{d^k}{dx^k}\chi_{2\lambda}(x) &= -\frac{\cos\beta}{s}\frac{d^k}{dx^k}\sin\left[s(\pi - x)\right] + O\left(\left|s\right|^{k-2} e^{|t|(\pi - x)}\right) & (22) \\ \frac{d^k}{dx^k}\chi_{1\lambda}(x) &= -\frac{\rho_{24}}{\rho_{34}}\cos\beta\cos\left[s\pi\right]\cos\left[sx\right] + O\left(\left|s\right|^{k-1} e^{|t|(\pi - x)}\right) & (23) \end{aligned}$$

as $|\lambda| \to \infty$ $(k = 0, 1)$. *Each of this asymptotic equalities hold uniformly for* x.

4. Asymptotic behaviour of eigenvalues and corresponding eigenfunctions

It is well-known from ordinary differential equation theory that the Wronskians $W[\phi_{1\lambda}, \chi_{1\lambda}]_x$ and $W[\phi_{2\lambda}, \chi_{2\lambda}]_x$ are independent of variable x. By using (2) and (3) we have

$$\begin{aligned} w_1(\lambda) &= \phi_1(0,\lambda)\frac{\partial\chi_1(0,\lambda)}{\partial x} - \chi_1(0,\lambda)\frac{\partial\phi_1(0,\lambda)}{\partial x} \\ &= \frac{\rho_{12}}{\rho_{34}}(\phi_2(0,\lambda)\frac{\partial\chi_2(0,\lambda)}{\partial x} - \chi_2(0,\lambda)\frac{\partial\phi_2(0,\lambda)}{\partial x}) \\ &= \frac{\rho_{12}}{\rho_{34}}w_2(\lambda) \end{aligned}$$

for each $\lambda \in \mathbb{C}$. It is convenient to introduce the notation

$$w(\lambda) := \rho_{34}w_1(\lambda) = \rho_{12}w_2(\lambda). \qquad (24)$$

Now by modifying the standard method we prove that all eigenvalues of the problem $(1) - (5)$ are real.

Theorem 3. *The eigenvalues of the boundary-value-transmission problem* $(1) - (5)$ *are real.*

PROOF.

Corollary 1. *Let $u(x)$ and $v(x)$ be eigenfunctions corresponding to distinct eigenvalues. Then they are orthogonal in the sense of the following equality*

$$\rho_{12}\int_{-\pi}^{0} u(x)v(x)dx + \rho_{34}\int_{0}^{\pi} u(x)v(x)dx = 0. \tag{25}$$

Since the Wronskians of $\phi_{2\lambda}(x)$ and $\chi_{2\lambda}(x)$ are independent of x, in particular, by putting $x = 1$ we have

$$\begin{aligned} w(\lambda) &= \phi_2(\pi,\lambda)\chi_2'(\pi,\lambda) - \phi_2'(\pi,\lambda)\chi_2(\pi,\lambda) \\ &= \cos\beta\phi_2(\pi,\lambda) + \sin\beta\phi_2'(\pi,\lambda). \end{aligned} \tag{26}$$

Let $\lambda = s^2$, $Ims = t$. By substituting (20) and (22) in (26) we obtain easily the following asymptotic representations
(i) If $\sin\beta \neq 0$ and $\sin\alpha \neq 0$, then

$$w(\lambda) = -\frac{\rho_{24}}{\rho_{12}}\sin\alpha\sin\beta\ s^2\sin^2[s\pi] + O\left(|s|\,e^{2\pi|t|}\right) \tag{27}$$

(ii) If $\sin\beta \neq 0$ and $\sin\alpha = 0$, then

$$w(\lambda) = \frac{\rho_{24}}{\rho_{12}}\cos\alpha\sin\beta\ s\cos[s\pi]\sin[s\pi] + O\left(e^{2\pi|t|}\right) \tag{28}$$

(iii) If $\sin\beta = 0$ and $\sin\alpha \neq 0$, then

$$w(\lambda) = \frac{\rho_{24}}{\rho_{12}}\sin\alpha\cos\beta\ s\sin[s\pi]\cos[s\pi] + O\left(e^{2\pi|t|}\right) \tag{29}$$

(iv) If $\sin\beta = 0$ and $\sin\alpha = 0$, then

$$w(\lambda) = -\frac{\rho_{24}}{\rho_{12}}\cos\beta\cos\alpha\cos^2[s\pi] + O\left(\frac{1}{|s|}e^{2\pi|t|}\right) \tag{30}$$

Now we are ready to derived the needed asymptotic formulas for eigenvalues and eigenfunctions.

Theorem 4. *The boundary-value-transmission problem* (1)-(5) *has an precisely numerable many real eigenvalues, whose behavior may be expressed by* $\{\lambda_{n,1}\}$ *with following asymptotic as* $n \to \infty$

(i) *If* $\sin\beta \neq 0$ *and* $\sin\alpha \neq 0$*,then*

$$s_{n,1} = (n - \frac{1}{2}) + O\left(\frac{1}{n}\right) \tag{31}$$

(ii) *If* $\sin\beta \neq 0$ *and* $\sin\alpha = 0$*, then*

$$s_{n,1} = \frac{n}{2} + O\left(\frac{1}{n}\right), \tag{32}$$

(iii) *If* $\sin\beta = 0$ *and* $\sin\alpha \neq 0$*, then*

$$s_{n,1} = \frac{n}{2} + O\left(\frac{1}{n}\right), \tag{33}$$

(iv) *If* $\sin\beta = 0$ *and* $\sin\alpha = 0$*, then*

$$s_{n,1} = \frac{n}{2} + O\left(\frac{1}{n}\right), \tag{34}$$

where $\lambda_{n,1} = s_{n,1}^2$.

PROOF.

Using this asymptotic expression of eigenvalues we can easily obtain the corresponding asymptotic expressions for eigenfunctions of the problem (1)-(5). Recalling that $\phi_{s_n}(x)$ is an eigenfunction according to the eigenvalue s_n, and by putting (31) in the (16)-(17) for $k = 0, 1$ and denoting the corresponding eigenfunction as

Theorem 5. ***(i)*** *If* $\sin\beta \neq 0$ *and* $\sin\alpha \neq 0$*,then*

$$\phi_{s_n}(x) = \sin\alpha\cos\left[(n - \frac{1}{2})\,(x + \pi)\right] + O\left(\frac{1}{n}\right)$$

(ii) *If* $\sin\beta \neq 0$ *and* $\sin\alpha = 0$*, then*

$$\phi_{s_n}(x) = \frac{\rho_{24}}{\rho_{12}}\sin\alpha\frac{n}{2}\sin\left[\frac{n}{2}\pi\right]\cos\left[\frac{n}{2}x\right] + O\,(1)$$

(iii) *If* $\sin\beta = 0$ *and* $\sin\alpha \neq 0$*, then*

$$\phi_{s_n}(x) = -\frac{2\cos\alpha}{n}\sin\left[\frac{n}{2}(x + \pi)\right] + O\left(\frac{1}{n^2}\right)$$

(ii) *If* $\sin\beta = 0$ *and* $\sin\alpha = 0$*, then*

$$\phi_{s_n}(x) \quad = \quad -\frac{\rho_{24}}{\rho_{12}}\cos\alpha\cos\left[\frac{n}{2}\pi\right]\cos\left[\frac{n}{2}x\right] + O\left(\frac{1}{n}\right)$$

as $|\lambda| \to \infty$ $(k = 0, 1)$*. Each of this asymptotic equalities hold uniformly for* x*.*

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